

ABSORBING ANGLES, STEINER MINIMAL TREES, AND ANTIPODALITY

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ABSTRACT. We give a new proof that a star $\{op_i : i = 1, \dots, k\}$ in a normed plane is a Steiner minimal tree of its vertices $\{o, p_1, \dots, p_k\}$ if and only if all angles formed by the edges at o are absorbing [Swanepoel, Networks **36** (2000), 104–113]. The proof is more conceptual and simpler than the original one.

We also find a new sufficient condition for higher-dimensional normed spaces to share this characterization. In particular, a star $\{op_i : i = 1, \dots, k\}$ in any CL-space is a Steiner minimal tree of its vertices $\{o, p_1, \dots, p_k\}$ if and only if all angles are absorbing, which in turn holds if and only if all distances between the normalizations $\frac{1}{\|p_i\|}p_i$ equal 2. CL-spaces include the mixed ℓ_1 and ℓ_∞ sum of finitely many copies of \mathbb{R} .

Keywords: Steiner minimal tree, absorbing angle, antipodality, face antipodality, Minkowski geometry.

1. INTRODUCTION

1.1. Minkowski geometry. Let \mathcal{M}^d denote a d -dimensional normed space (or *Minkowski space*) with origin o , i.e., \mathbb{R}^d equipped with a norm $\|\cdot\|$. We call an \mathcal{M}^2 a *Minkowski plane*. Denote the *unit ball* by $B = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$. The *dual* \mathcal{M}_*^d of \mathcal{M}^d is \mathbb{R}^d equipped with the *dual norm*

$$\|x\|_* := \max_{\|y\| \leq 1} \langle x, y \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on \mathbb{R}^d . The *dual unit ball*

$$B_* = \{x \in \mathbb{R}^d : \langle x, y \rangle \leq 1 \ \forall y \in B\}$$

is also known as the *polar body* of B .

For example, the d -dimensional Minkowski spaces ℓ_1^d and ℓ_∞^d are duals of each other, where ℓ_1^d has the norm $\|(x_1, \dots, x_d)\|_1 := \sum_{i=1}^d |x_i|$ and ℓ_∞^d has the norm $\|(x_1, \dots, x_d)\|_\infty := \max\{|x_i| : i = 1, \dots, d\}$.

A vector $x_* \in \mathcal{M}_*^d$ is *dual* to $x \in \mathcal{M}^d$, $x \neq o$, if $\|x_*\|_* = 1$ and $\langle x_*, x \rangle = \|x\|$, i.e., x_* is a dual unit vector that attains its norm at x . In this case the hyperplane $\{x \in \mathbb{R}^d : \langle x_*, x \rangle = 1\}$ supports the unit ball at $\frac{1}{\|x\|}x$. Any hyperplane supporting the unit ball at $\frac{1}{\|x\|}x$ is given in this way by some x_* dual to x . A unit vector $v \in \mathcal{M}^d$ is a *regular direction* if there is only one hyperplane that supports B at v . Note that the norm function $f(x) := \|x\|$ is differentiable at $p \neq o$ if and only if $\frac{1}{\|p\|}p$ is a regular direction, and then the gradient $\nabla f(p)$ is the unique vector in \mathcal{M}_*^d dual to p .

The *exposed face* of the unit ball B defined by a unit vector $a^* \in \mathcal{M}_*^d$ is

$$[a^*] := \{a \in B : \langle a, a^* \rangle = 1\}.$$

Similarly, a unit vector $a \in \mathcal{M}^d$ defines an exposed face $[a]^*$ of B^* . If B is a polytope then all faces are exposed, and each face F of B corresponds to a face F^* of B^* as follows:

$$F^* := \{a^* \in B^* : \langle a, a^* \rangle = 1 \text{ for all } a \in F\}.$$

Research supported by a grant from an agreement between the Deutsche Forschungsgemeinschaft in Germany and the National Research Foundation in South Africa.

1.2. Trees. Let $S \subset \mathcal{M}^d$ be a finite, non-empty set of points. A *spanning tree* T of S is an acyclic connected graph with vertex set S . Denote its edge set by $E(T)$. A *Steiner tree* T of S is a spanning tree of some finite $V \subset \mathcal{M}^d$ such that $S \subseteq V$ and such that the degree of each vertex in $V \setminus S$ is at least 3. The vertices in S are the *terminals* of the Steiner tree T , and the vertices in $V \setminus S$ the *Steiner points* of T . The *length* of a tree T in \mathcal{M}^d is

$$\ell(T) := \sum_{xy \in E(T)} \|x - y\|.$$

A *Steiner minimal tree* (SMT) of S is a Steiner tree of S of smallest length. The requirement that Steiner points have degree at least 3 is for technical convenience, since Steiner points of degree at most 2 can easily be eliminated using the triangle inequality without making the tree longer. It is easily seen that the number of Steiner points is at most $\#S - 2$. It then follows by a simple compactness argument that any non-empty finite S has a SMT.

A *star* with center s is a tree in which the vertex s is joined to all other vertices. If $s \in S$ has neighbors $s_1, \dots, s_k \in V$ in some SMT, then clearly the star joining s to each s_i , $i = 1, \dots, k$, is a SMT of $\{s, s_1, \dots, s_k\}$. Thus, to characterize the neighborhoods of terminals in SMTs, it is sufficient to characterize SMTs which are stars with the center a terminal. This is the intent of Theorem 1 below.

1.3. Angles. An angle $\angle x_1 x_0 x_2$ in \mathcal{M}^d is *absorbing* if the function

$$x \mapsto \|x - x_0\| + \|x - x_1\| + \|x - x_2\|$$

attains its minimum at x_0 . Thus $\angle x_1 x_0 x_2$ is absorbing if and only if the star $\{x_0 x_1, x_0 x_2\}$ is a SMT of $\{x_0, x_1, x_2\}$.

Lemma 1. *Let a and b be unit vectors in \mathcal{M}^d . Then the following are equivalent:*

- (1) $\angle aob$ is absorbing.
- (2) There exist unit vectors a^* and b^* in \mathcal{M}_*^d such that $\langle a^*, a \rangle = \langle b^*, b \rangle = 1$ and $\|a^* + b^*\|^* \leq 1$.
- (3) The exposed faces $[a]^*$ and $[-b]^* = -[b]^*$ of the dual unit ball are at distance ≤ 1 .

Proof. (1) \Leftrightarrow (2) is part of Lemma 5.4 in [5]. (2) \Leftrightarrow (3) is trivial. \square

In particular, this is a property of the angle alone:

Corollary 1. *If y_i is a point on the ray $\overrightarrow{x_0 x_i}$, $i = 1, 2$, then $\{x_0 x_1, x_0 x_2\}$ is a SMT of $\{x_0, x_1, x_2\}$ if and only if $\{x_0 y_1, x_0 y_2\}$ is a SMT of $\{x_0, y_1, y_2\}$. [5, Proposition 3.3].*

Furthermore, an angle containing an absorbing angle is itself absorbing.

See also Proposition 3.3 and Lemma 5.4 of [5].

1.4. The planar case. We are now able to formulate the first result. In any Minkowski space, all angles made by two incident edges of a SMT are clearly absorbing. (For angles in a minimal spanning tree even more is true [4].) Remarkably, as shown in [8], for a Minkowski plane the condition that all angles are absorbing is also sufficient for a star to be a SMT of its vertices.

Theorem 1. *Let $p_1, \dots, p_k \neq o$ be points in a Minkowski plane \mathcal{M}^2 . Then the star joining each p_i to o is a SMT of $\{o, p_1, \dots, p_k\}$ if (and only if) all angles $\angle p_i o p_j$, $i \neq j$, are absorbing.*

This result is used in [8] to show that the maximum degree of a vertex in a SMT in a Minkowski plane is 6, with equality only if the unit ball is an affine regular hexagon; for all other planes the maximum is 4 if there exist supplementary absorbing angles, and 3 otherwise. The proof given in [8] employs a long case analysis. The new proof presented in Section 2 is more conceptual.

1.5. Antipodality and higher dimensions. Theorem 1 does not hold anymore in Minkowski spaces of dimension at least 3. For example, let the unit ball be the projection of a $(d+1)$ -cube along a diagonal. (When $d = 3$, this is the rhombic dodecahedron.) In this Minkowski space, the star joining o to all $2^{d+1} - 2$ vertices of the unit ball is not a SMT of these vertices if $d \geq 3$, despite all the angles being absorbing [9]. However, Theorem 1 extends to both ℓ_1^d and ℓ_∞^d . Our second result is a generalization of this fact.

We first introduce some more notions, involving antipodality. Two boundary points of the unit ball B are *antipodal* if there exist distinct parallel hyperplanes supporting the two points. Equivalently, unit vectors a and b are antipodal if and only if $\|a - b\| = 2$.

Lemma 2. *If a and b are antipodal unit vectors in a Minkowski space, then $\angle aob$ is an absorbing angle.*

Proof. Since the union of the segments oa and ob form a shortest path from a to b , these two segments form a SMT of $\{o, a, b\}$, hence $\angle aob$ is absorbing. \square

The converse of the above lemma is not necessarily true, as the Euclidean norm shows. We call the unit ball of a Minkowski space *Steiner antipodal* if two points a and b on the boundary of the unit ball are antipodal whenever $\angle aob$ is absorbing.

Theorem 2. *Consider the following properties of a set $\{p_1, \dots, p_k\}$ of unit vectors in a Minkowski space \mathcal{M}^d .*

(4) *All angles $\angle p_i o p_j$ are absorbing.*

(5) *All distances $\|p_i - p_j\| = 2$.*

(6) *The star $\bigcup_{i=1}^k [o, p_i]$ is a SMT of $\{p_1, \dots, p_k\}$.*

(7) *The star $\bigcup_{i=1}^k [o, p_i]$ is a SMT of $\{o, p_1, \dots, p_k\}$.*

Then the implications $(5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (4)$ hold. Furthermore, (4) to (7) are equivalent if, and only if, the norm is Steiner antipodal.

Proof. $(6) \Rightarrow (7)$ is true, since all Steiner trees of $\{o, p_1, \dots, p_k\}$ are also Steiner trees of $\{p_1, \dots, p_k\}$, by considering o to be a Steiner point.

$(7) \Rightarrow (4)$ holds, since if any star $[o, p_i] \cup [o, p_j]$ can be shortened, it would also shorten the star $\bigcup_{i=1}^k [o, p_i]$.

The implication $(4) \Rightarrow (5)$ is equivalent to the definition of Steiner antipodality.

This leaves $(5) \Rightarrow (6)$. Note that the given star is a Steiner tree of length k . It is sufficient to show that all Steiner trees have length $\geq k$. However, note that the open unit balls centered at the p_i are pairwise disjoint, since $\|p_i - p_j\| = 2$ for distinct $i \neq j$. Any Steiner tree will have to join each p_i to the boundary of the unit ball with centre p_i . The part of the Steiner tree inside this ball must therefore have length at least 1. It follows that the length of any Steiner tree must be at least k . \square

In order to apply this result, we need a characterization of Steiner antipodal norms in terms of duality.

Proposition 1. *The following are equivalent in any Minkowski space \mathcal{M}^d :*

(8) *The norm is Steiner antipodal.*

(9) *The unit ball is a polytope and any two disjoint faces of the dual unit ball are at distance > 1 .*

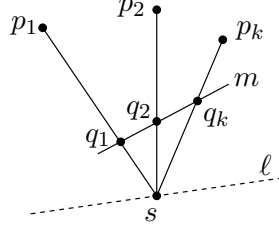


FIGURE 1.

Proof. (8) \Leftarrow (9) is immediate from the definition of Steiner antipodality and Lemma 1. (8) \Rightarrow (9) follows upon noting that if a convex body is not a polytope, then there are disjoint exposed faces that are arbitrarily close to each other. \square

A Minkowski space is a *CL-space* if for every maximal proper face F of the unit ball B we have $B = \text{conv}(F \cup (-F))$. It is easily seen from finite dimensionality that the unit ball of a CL-space is a polytope. CL-spaces were introduced by R. E. Fullerton (see [7]), although the notion has been studied before by Hanner [1], who proved that the unit balls of CL-spaces are $\{0, 1\}$ -polytopes. McGregor [6] showed that CL-spaces are exactly those spaces with numerical index 1. What is important for our purposes is that CL-spaces turn out to be Steiner antipodal.

Hanner [1] identified an important subclass of CL-spaces, namely those that can be built up from the one-dimensional space \mathbb{R} using ℓ_1 -sums and ℓ_∞ -sums. For two Minkowski spaces \mathcal{M} and \mathcal{N} of dimension d and e we define their ℓ_1 -sum $\mathcal{M} \oplus_1 \mathcal{N}$ and ℓ_∞ -sum $\mathcal{M} \oplus_\infty \mathcal{N}$ to be the Minkowski spaces on \mathbb{R}^{d+e} with norms $\|(x, y)\|_1 = \|x\| + \|y\|$ and $\|(x, y)\|_\infty = \max\{\|x\|, \|y\|\}$. Note that the unit ball of $\mathcal{M} \oplus_1 \mathcal{N}$ is the convex hull of the unit ball of \mathcal{M} when embedded as $\mathcal{M} \oplus \{o\}$ and the unit ball of \mathcal{N} when embedded as $\{o\} \oplus \mathcal{N}$. The unit ball of $\mathcal{M} \oplus_\infty \mathcal{N}$ is the Cartesian product of the unit balls of \mathcal{M} and \mathcal{N} . The unit balls of these spaces are called *Hanner polytopes*. We thus introduce the name *Hanner space* for these spaces. For more information see [1, 3, 2, 7].

We summarize the above discussion as follows.

Proposition 2. *All Hanner spaces are CL-spaces. All CL-spaces are Steiner antipodal.*

Proof. It is clear and well-known that Hanner spaces are CL-spaces (see, e.g., [7]).

It is also well-known that the dual of a CL-space is a CL-space as well [6]. To prove the second part of the proposition, it is by Proposition 1 sufficient to show that any two disjoint faces F and G of the unit ball B are at distance > 1 . Suppose that F is contained in the facet F' . Then all vertices of B disjoint from F must lie in the opposite facet $-F'$. It follows that $G \subseteq -F'$, and F and G are therefore at distance 2. \square

2. PROOF OF THEOREM 1

Lemma 3. *Let ℓ be a line passing through a Steiner point s of a SMT T in a Minkowski plane \mathcal{M}^2 . Assume that ℓ is parallel to a regular direction. Then T has edges incident to s in both open half planes bounded by ℓ .*

Proof. Without any assumption on ℓ , the edges incident to s cannot all lie in the same open half plane bounded by ℓ . Indeed, such a tree can be shortened as follows (Fig. 1). Let some line m intersect the interior of each edge sp_i , $i = 1, \dots, k$, in q_i , say. Remove edges sp_1 , sp_k , and sq_i , $i = 2, \dots, k-1$, and add edges p_1q_2 , q_iq_{i+1} , $2 \leq i \leq k-2$, and $q_{k-1}p_k$, to obtain a new Steiner tree T' , without the Steiner point s , but with new Steiner points q_i , $2 \leq i \leq k-1$. By the triangle inequality, $\ell(T) - \ell(T') \geq \sum_{i=2}^{k-1} \|sq_i\| > 0$, contradicting the minimality of T .

We now assume that ℓ is parallel to a regular direction. It is sufficient to show that sp_1 and sp_k cannot be opposite edges both on ℓ , with all other sp_i , $2 \leq i \leq k-1$, on the same side of ℓ (Fig. 2). Let s_2 be a variable point on sp_2 with $\|s_2 - s\|$ small. Denote the intersection of

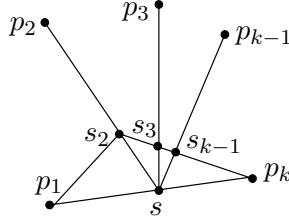


FIGURE 2.

$s_2 p_k$ and $s p_i$ by s_i , $i = 3, \dots, k-1$. Change the Steiner tree T as follows. Remove edges $p_1 s$, $p_k s$ and $s_i s$, $i = 2, \dots, k-1$, and add edges $p_1 s_2$ and $s_2 p_k$. This removes the Steiner point s and introduces new Steiner points s_2, \dots, s_{k-1} . Denoting the new tree by T' , it follows that the length changes by

$$\begin{aligned} \ell(T') - \ell(T) &= \|s_2 - p_1\| + \|s_2 - p_k\| - \|s - p_1\| - \|s - p_k\| - \sum_{i=2}^{k-1} \|s - s_i\| \\ &\leq \|(s - p_1) + (s_2 - s)\| - \|s - p_1\| \\ &\quad + \|(s - p_k) + (s_2 - s)\| - \|s - p_k\| - \|s_2 - s\|. \end{aligned}$$

Since $s - p_1$ and $s - p_k$ are parallel to a regular direction, the norm is differentiable at both points, i.e.,

$$\lim_{s_2 \rightarrow s} \frac{\|(s - p_1) + (s_2 - s)\| - \|s - p_1\|}{\|s_2 - s\|} = \langle u_*, s_2 - s \rangle$$

and

$$\lim_{s_2 \rightarrow s} \frac{\|(s - p_k) + (s_2 - s)\| - \|s - p_k\|}{\|s_2 - s\|} = \langle -u_*, s_2 - s \rangle.$$

(Since $s - p_1$ and $s - p_k$ are in opposite directions, their duals are opposite in sign.) It follows that

$$\begin{aligned} \ell(T') - \ell(T) &\leq \langle u_*, s_2 - s \rangle + \langle -u_*, s_2 - s \rangle + o(\|s_2 - s\|) - \|s_2 - s\| \\ &= o(\|s_2 - s\|) - \|s_2 - s\|, \end{aligned}$$

which is negative if $\|s_2 - s\|$ is sufficiently small. Then $\ell(T') < \ell(T)$, a contradiction. \square

Proof of Theorem 1. Without loss of generality, the segments op_1, \dots, op_k are ordered around o . Assume that all angles $\angle p_i op_j$ are absorbing. We start off with an arbitrary SMT of $\{o, p_1, \dots, p_k\}$ and modify it in two steps without increasing the length. In Step 1 we eliminate all Steiner points in the interiors of the angles $\angle p_i op_{i+1}$. In Step 2 we eliminate all edges between vertices on different rays $\overrightarrow{op_i}$. The edges of the final SMT are then all contained in the union of the segments op_i , $i = 1, \dots, k$. This tree cannot have Steiner points, and so has to be the star with centre o . This concludes the proof.

Step 1: For each angle $\angle p_i op_{i+1}$ (where we let $k+1 \equiv 0$), choose a regular direction r_i not contained in the (closed) angle. Choose $\tilde{p}_i \in \overrightarrow{op_i}$ and $\tilde{q}_i \in \overrightarrow{op_{i+1}}$ such that $\tilde{p}_i \tilde{q}_i$ is parallel to r_i (Fig. 3). For each point s in the interior of $\angle p_i op_j$, write $s = \alpha p + \beta q$ (uniquely, and then, moreover, $\alpha, \beta > 0$) and define the *measure* of s to be

$$|s| := \alpha + \beta.$$

Define the *measure* $|T|$ of any Steiner tree T of $\{o, p_1, \dots, p_k\}$ to be the sum of the measures of all Steiner points of T not on any ray $\overrightarrow{op_i}$. Let

$$\mu = \inf\{|T| : T \text{ is a SMT of } \{o, p_1, \dots, p_k\}\}.$$

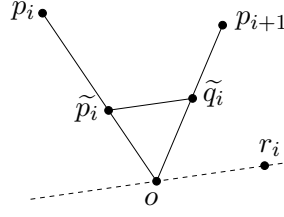


FIGURE 3.

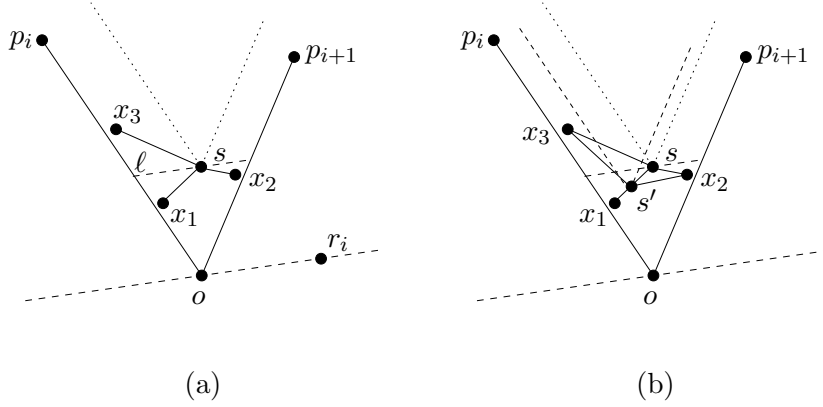


FIGURE 4.

Let T_n be a sequence of SMTs of $\{o, p_1, \dots, p_k\}$ with $\lim_{n \rightarrow \infty} |T_n| = \mu$. Since there are only finitely many combinatorial types of Steiner trees on a set of $k+1$ points, we may, by passing to a subsequence, assume without loss of generality that all T_n have the same combinatorial type with Steiner points $s_1^{(n)}, \dots, s_m^{(n)}$, say. By taking further subsequences, we may assume that each sequence of Steiner points converge, say $s_i^{(n)} \rightarrow s_i$, $i = 1, \dots, m$. In the limit we obtain a Steiner tree T_0 with $\ell(T_0) = \lim_{n \rightarrow \infty} \ell(T_n)$, hence T_0 is a SMT. Also, $|T_0| \leq \lim_{n \rightarrow \infty} |T_n|$, since the measure of a Steiner point is continuous in the interior of an angle, hence $\lim_{n \rightarrow \infty} |s_i^{(n)}| = |s_i|$ if s_i is still in the interior of the same angle, otherwise $\lim_{n \rightarrow \infty} |s_i^{(n)}| \geq |s_i| = 0$ if s_i is on one of the rays $\overrightarrow{op_j}$. Therefore, $|T_0| = \mu$. It remains to show that $\mu = 0$, since this will imply that T_0 does not have any Steiner point in the interior of an angle.

Suppose that $\mu > 0$. We obtain a contradiction by constructing a SMT T' with $|T'| < \mu$. Let s be a Steiner point of T_0 in the interior of $\angle p_i o p_{i+1}$, say (Fig. 4(a)). Without loss of generality there is no point of T_0 in the translated angle $s + \angle p_i o p_{i+1}$, since such a point is necessarily another Steiner point s' and we may then repeatedly choose a new Steiner point s'' in $s' + \angle p_i o p_{i+1}$, until this procedure halts.

Let ℓ be the line through s parallel to r_i . The points on ℓ in the interior of $\angle p_i o p_{i+1}$ all have the same measure, and the points on the same side of ℓ as o have smaller measure. By Lemma 3 there is an edge sx_1 incident to s on the same side of ℓ as o . There are at least two more edges sx_2 and sx_3 . Since not all edges are in an open half plane bounded by a line through s , we may choose x_2 and x_3 such that the angle $\angle x_2 s x_3$ contains the translated angle $s + \angle p_1 o p_2$ in its interior (with s excluded). It follows that there is a point s' on sx_1 sufficiently close to s such that $\angle x_2 s' x_3$ contains the translate $s' + \angle p_1 o p_2$, and so is still absorbing (Figure 4(b)). We may therefore replace the edges sx_2 , sx_3 and ss' by $s'x_2$ and $s'x_3$ without lengthening T_0 , to obtain a new SMT T' . However, $|s'| < |s|$, hence $|T'| < |T| = \mu$, which gives the required contradiction.

Step 2: Note that for any absorbing angle $\angle p_i o p_j$,

$$\|p_i - o\| + \|p_j - o\| + \|o - o\| \leq \|p_i - p_j\| + \|p_j - p_j\| + \|o - p_j\|,$$

i.e., $\|p_i - p_j\| \geq \|p_i\|$.

Suppose that the SMT T has an edge between two points on different segments, say between q_i on op_i and q_j on op_j . Without loss of generality, the unique path in T from o to q_i passes through q_j (otherwise interchange q_i and q_j). Since $\angle q_i o q_j$ is absorbing, $\|q_i - q_j\| \geq \|q_i\|$. We can then replace the edge $q_i q_j$ by $o q_j$, without losing connectivity and without lengthening T . This process may be repeated until all edges are on the segments op_i , which finishes Step 2. \square

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